The Uncertainty Relation and its Applications in Cryptography

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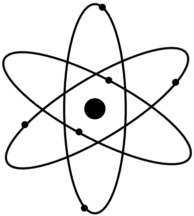
Group of Applied Physics, University of Geneva

[PRL **106**, 110506 (2011)] and [arXiv: 1103.4130]

QCrypt, Zürich, September 2011

Classical Observers



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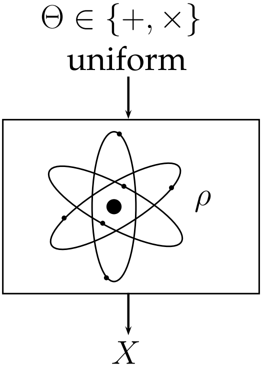

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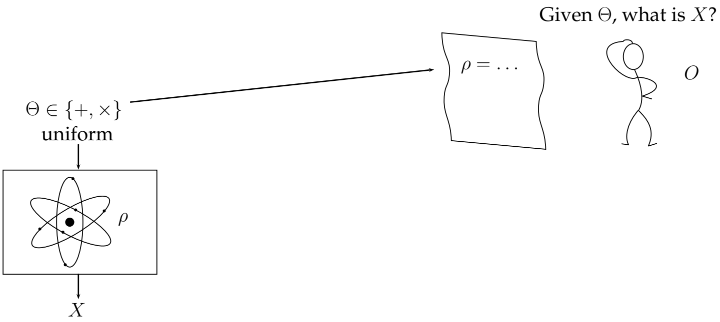

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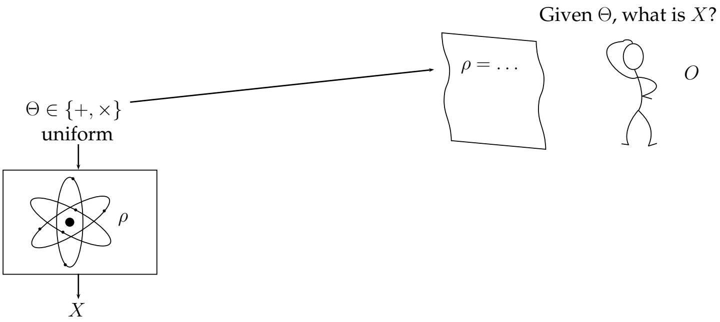

- Heisenberg uncertainty principle:
  Observer cannot predict outcome of both measurements.

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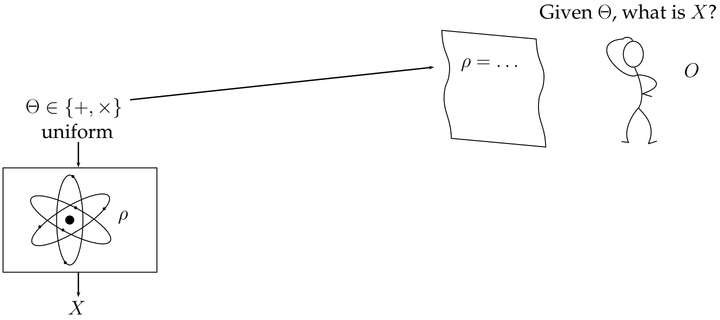
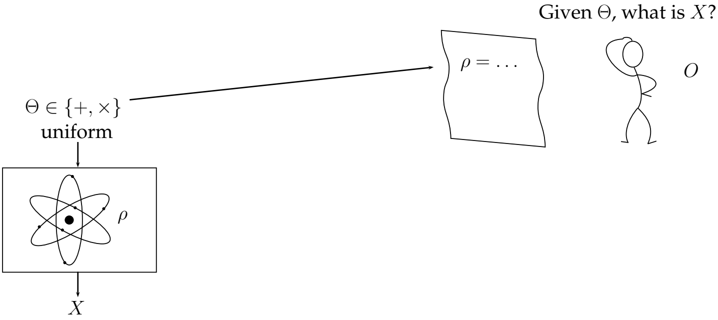


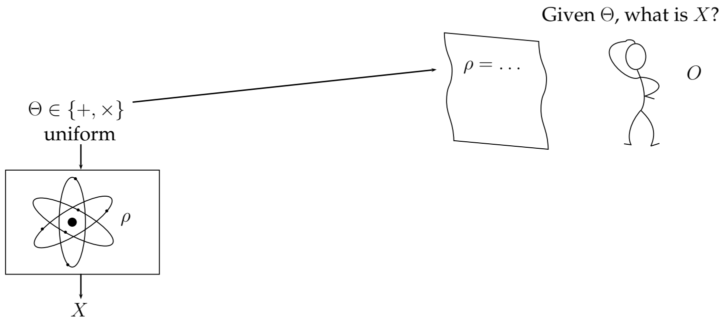
- Heisenberg uncertainty principle:
  Observer cannot predict outcome of both measurements.
- For qubits, unbiased bases and ρ eigenstate of '$+$':

$$H(X|O,\Theta) = \frac{1}{2}H(X|O,\Theta{=}+) + \frac{1}{2}H(X|O,\Theta{=}\times) = 0 + \frac{1}{2}.$$



Deutsch, Maassen/Uffink 1988

For unbiased bases and qubits: $H(X|O,\Theta) \geq \frac{1}{2}$.



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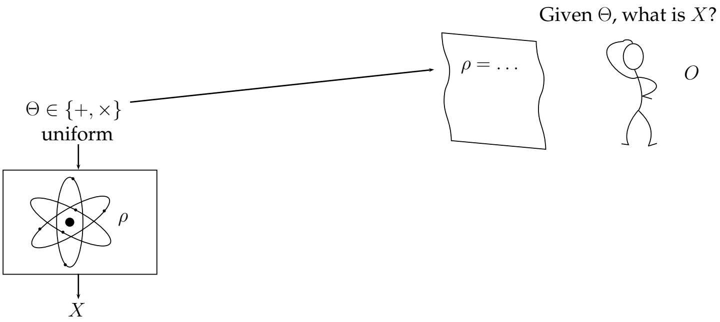
For unbiased bases: $H(X|O,\Theta) \geq \frac{1}{2}\log_2 d$.

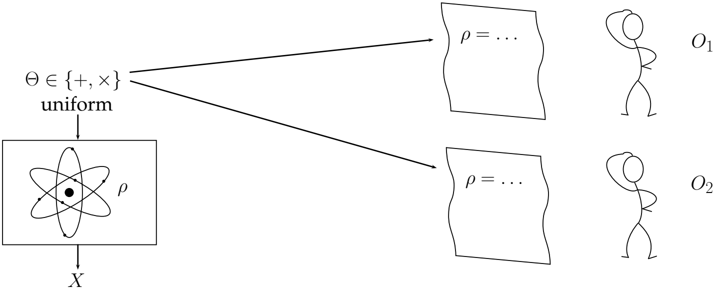


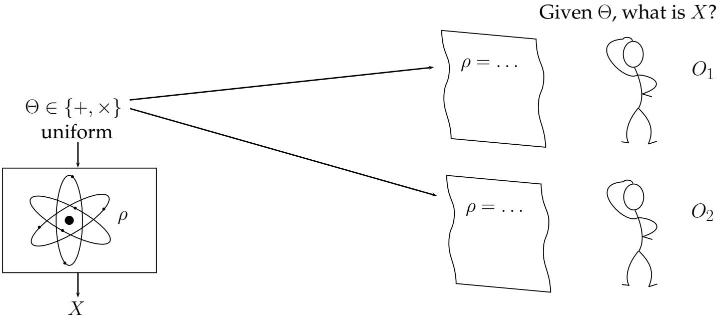
Deutsch, Maassen/Uffink 1988

$$H(X|O,\Theta) \geq \frac{1}{2}\log_2\frac{1}{c}, \quad c = \max_{x,y}\left|\langle x|y\rangle\right|^2,$$

where $|x\rangle$ and $|y\rangle$ are eigenvectors of the two measurements.

Given Θ, what is X?

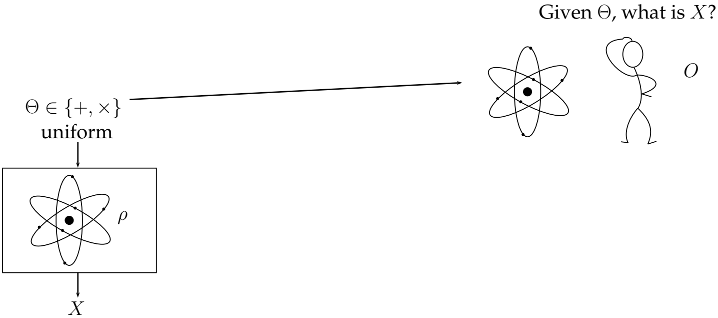


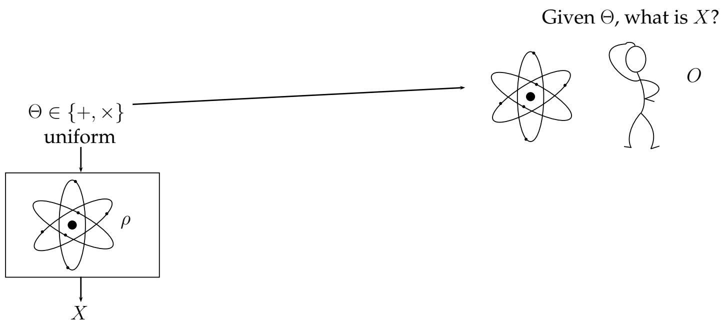


Deutsch, Maassen/Uffink 1988

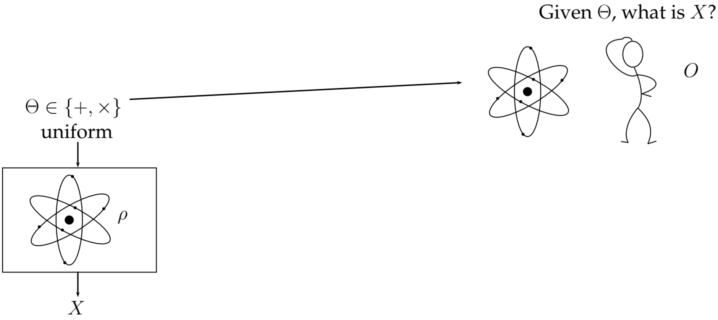
$$H(X|O_1, \Theta) + H(X|O_2, \Theta) \geq \log_2 \frac{1}{c} \, .$$

Quantum Observers





- Consider joint state ρ_{AO}.



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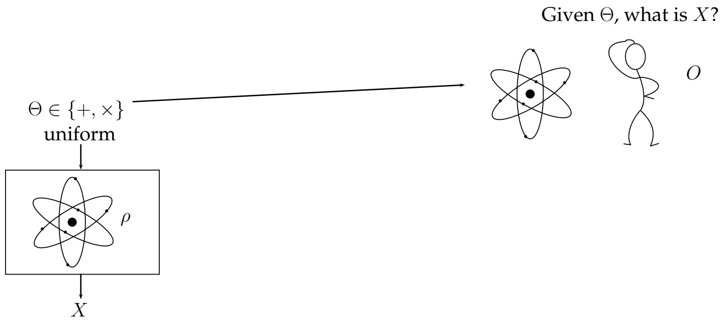
- If $\rho_{AO} = |\psi\rangle\langle\psi|$ is fully entangled, then $H(X|O, \Theta) = 0$. (Observer chooses measurement on O — depending on Θ — to get perfect correlation with X.)



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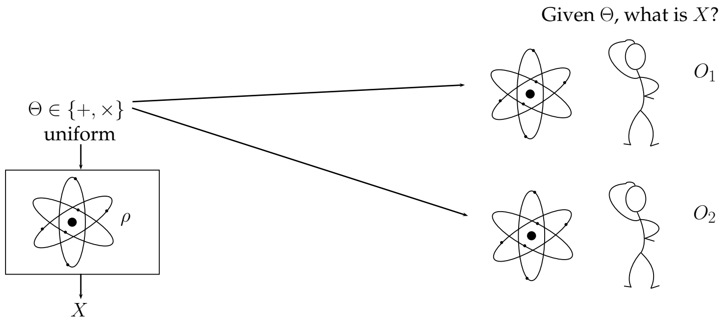
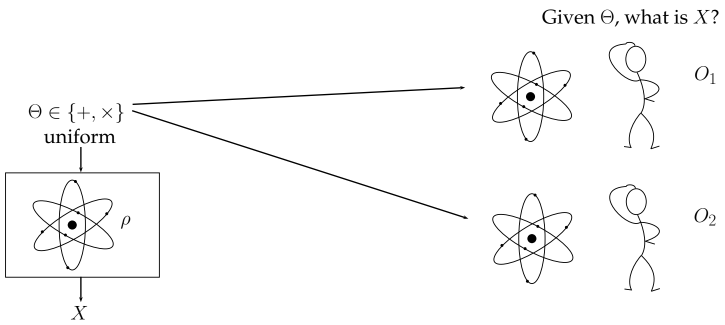
- If $\rho_{AO} = |\psi\rangle\langle\psi|$ is fully entangled, then $H(X|O, \Theta) = 0$. (Observer chooses measurement on O — depending on Θ — to get perfect correlation with X.)
- No uncertainty relation here!

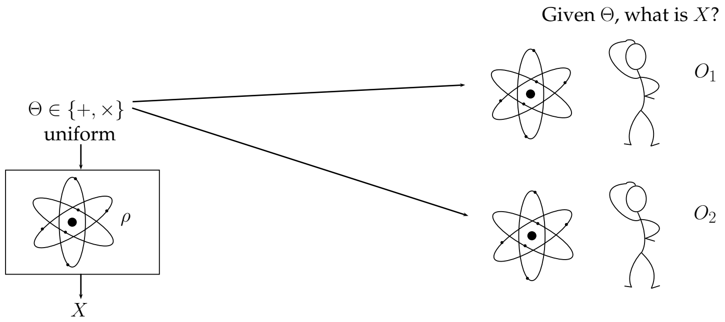


Can monogamy of entanglement help?

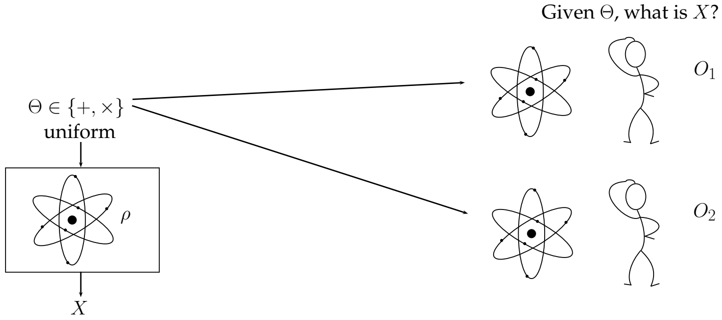


Berta et al. 2010, Coles et al. 2011

$$H(X|O_1, \Theta) + H(X|O_2, \Theta) \geq \log_2 \frac{1}{c}\,.$$



- (Conditional) von Neumann entropies have many applications.



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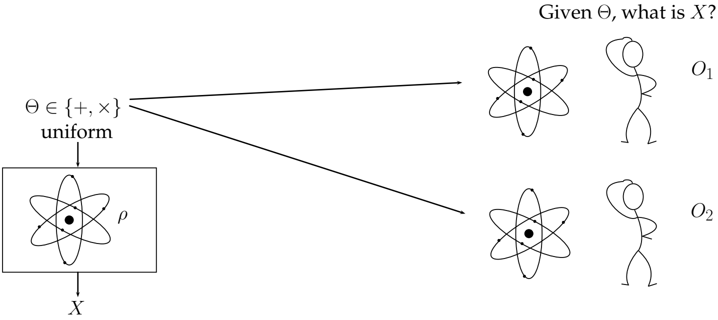
- In some settings, e.g. in cryptography, other entropies are more relevant.



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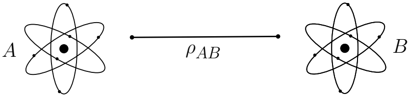
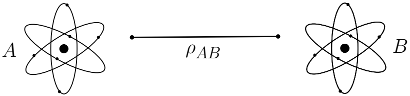
- We now extend the uncertainty relation to smooth entropies.



- A perfect observer B of a quantum system A is described by a state $|\psi\rangle\langle\psi|_{AB'} \otimes \sigma_{B''}$, where $|\psi\rangle$ is fully entangled.



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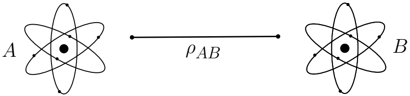
- Proximity to a perfect observer:

$$F_{\text{perfect}}(A|B) = \max_{B \to B'B''} \max_{\sigma} F\left(\rho_{AB'B''}, |\psi\rangle\langle\psi|_{AB'} \otimes \sigma_{B''}\right).$$



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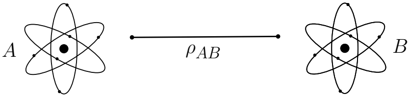
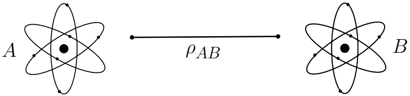
- Min-Entropy: $H_{\min}(A|B) := -\log F^2_{\text{perfect}}(A|B)$.



- An ignorant observer B of a quantum system A is described by a state $\omega_A \otimes \sigma_B$, where ω_A is completely mixed.



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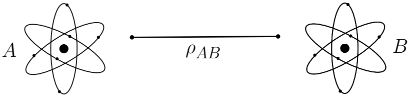

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- We optimize entropies over a ball of close states, $\mathcal{B}^{\varepsilon}(\rho)$.

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- For classical $A = X$, it characterizes the extractable independent randomness:

Renner 2005, MT/Schaffner/Smith/Renner 2011

The number of random bits — independent of a (quantum) memory B — that can be extracted from X is

$$\ell_{\text{extr}} \approx H_{\min}^{\varepsilon}(X|B) \,.$$

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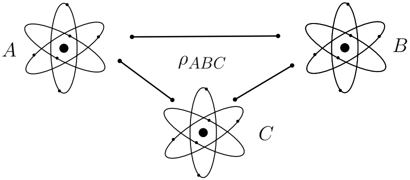
- For classical $A = X$, it characterizes the encoding length for data reconciliation:

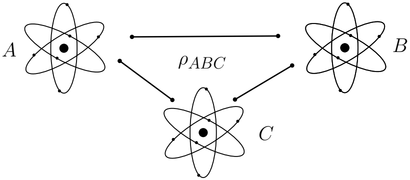
Renner/Renes 2010 [arXiv:1008.0452]

The number of bits needed to reconstruct X from a (quantum) memory B is

$$\ell_{\text{enc}} \approx H^{\varepsilon}_{\max}(X|B) \, .$$

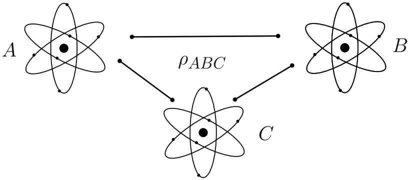
Duality





- Quantum mechanics implies uniqueness of perfect observer due to monogamy of entanglement. Moreover,

$$F_{\text{perfect}}(A|B) \leq F_{\text{ignorant}}(A|C)\,.$$



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- In terms of entropies [König/Renner/Schafner, 2008]:

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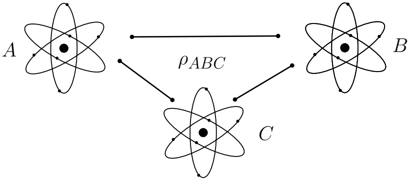

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- And smooth entropies [MT/Colbeck/Renner, 2010]:

$$H^{\varepsilon}_{\min}(A|B) + H^{\varepsilon}_{\max}(A|C) \geq 0 .$$

The uncertainty relation for smooth entropies:

MT/Renner 2011

For any state $\rho_{AO_1O_2}$, $\varepsilon \geq 0$ and POVMs $\{M_x\}$ and $\{N_y\}$ on A:

$$H^{\varepsilon}_{\min}(X|O_1, \Theta) + H^{\varepsilon}_{\max}(X|O_2, \Theta) \geq \log_2 \frac{1}{c},$$

$$c = \max_{x,y} \left\| \sqrt{M_x}\sqrt{N_y} \right\|^2_{\infty}.$$

- Overlap is $c = \max_{x,y} \left|\langle x|y\rangle\right|^2$ for projective measurements.

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- This implies previous results for the von Neumann entropy due to asymptotic equipartition [MT/Colbeck/Renner, 2009]

$$\frac{1}{n} H^{\varepsilon}_{\min/\max}(A^n|B^n) \xrightarrow{n\to\infty,\,\varepsilon\to 0} H(A|B).$$

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- Operational quantities $\Longrightarrow$ Applications in cryptography.

- We consider the entanglement-based Bennett-Brassard 1984 protocol [Bennett/Brassard/Mermin, 1992]

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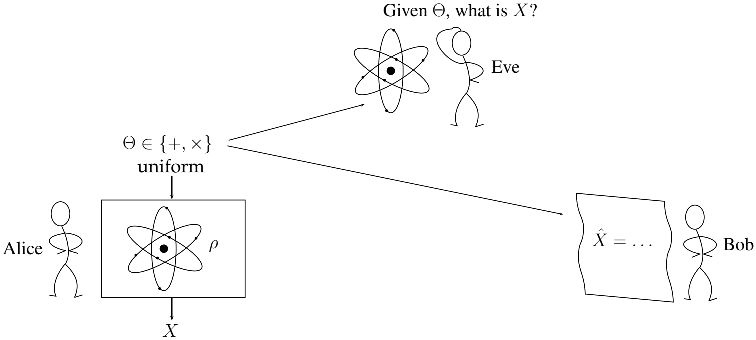
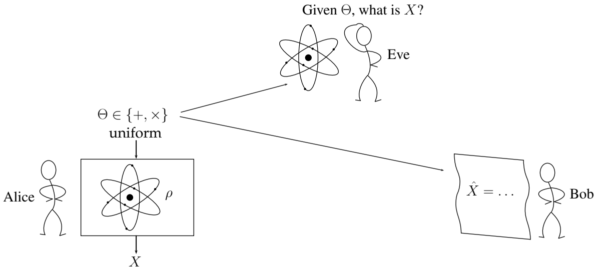
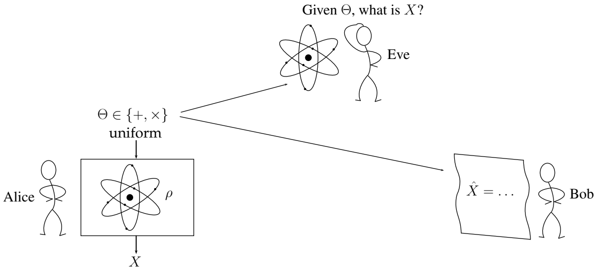
- The situation after Bob measured and holds an estimate $\hat{X}$ of X looks as follows:





- Overlap: $\log_2 \frac{1}{c} = 1$ per bit. (qubits and unbiased bases)

Security Proof Sketch



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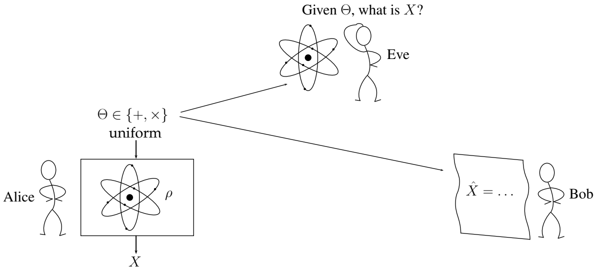

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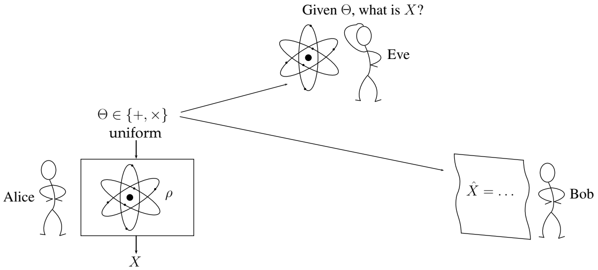
- Secret key:

$$\ell_{\mathrm{sec}} \gtrsim \ell_{\mathrm{extr}} - \ell_{\mathrm{enc}}$$



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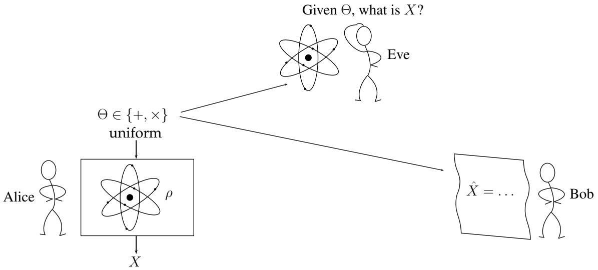

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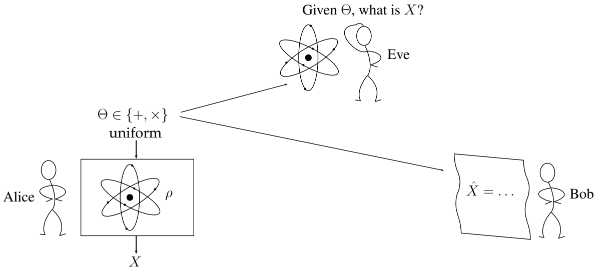

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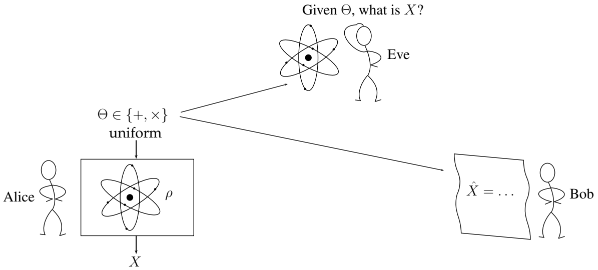

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- Parameter esimation: $\lambda = \frac{1}{k}|X^k \oplus \hat{X}^k|$.
- Then, estimate $H_{\max}^{\varepsilon}(X^n|\hat{X}^n) \lesssim nh(\lambda)$.

The extractable ϵ-secure key per block of size $N = n + k$ is

$$\ell^{\epsilon} \leq n\big(1 - h(Q_{\text{tol}} + \mu)\big) - 3\log(3/\epsilon) - \text{leak}_{\text{EC}}$$

- $\mu \approx \sqrt{1/k \cdot \ln(1/\epsilon)}$ is the statistical deviation from the tolerated channel noise, Q_{tol}.
- $\text{leak}_{\text{EC}} \approx nh(Q_{\text{tol}})$ is the information about the key leaked during error correction.
- The achievable key rate, ℓ/N, deviates from its optimal asymptotic value, $1 - 2h(Q)$, only by (unavoidable) terms that are due to finite statistics.

- The improved finite key bounds are due to the simplicity of the proof via the uncertainty relation.
  - Tomography of single quantum systems is unnecessary. Instead, the min-entropy of X^n is bounded directly.
  - Security against general attacks comes for free — no De Finetti or Post-Selection necessary.
- This proof technique can be applied to other problems in 3-party quantum cryptography.
- As pointed out by Hayashi/Tsurumaru [`arXiv:1107.0589`], the key rates can be improved if we allow a dynamic protocol that chooses a different ℓ in each run.

- The smooth entropies and uncertainty relation have been generalized to von Neumann algebras. [Berta/Furrer/Scholz, `arXiv: 1107.5460`].
- It was shown that the (effective) overlap of two measurements can be bounded by the CHSH violation that can be achieved with them. [Hänggi/MT, `arXiv: 1108.5349`]
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