Lower Dimensional Sections of Qutrit State Space using 3-Dimensional Vectors

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1. Introduction

The qutrit having three internal states comes next in complexity after qubit as a resource for quantum information processing. A qutrit state density matrix is of order 3 and depends on 8 parameters. Whereas the qubit density matrix can be easily visualized using Bloch sphere representation of its states, at the same time this simplicity is unavailable for the 8-dimensional state space of a qutrit. Earlier work tried to capture the complexity of the Qutrit State Space (QSS) by studying its 2 and 3 dimensional sections. Recently an alternative approach to study the QSS using 3 dimensional vectors has become available [1]. In this work we use them to study 2, 3, and other higher order sections.

2. Earlier Work

There have been many attempts at visualizing the QSS [2-7].

(i) Using all 8 Gell-Mann matrices, which form a complete set for expressing 3 X 3 SU(3) matrices

(ii) Using 6 Gell-Mann matrices supplemented by different matrices in place of two diagonal ones

(iii) 3-dimensional Bloch matrices and their principal minors

The sections of the 8-dimensional QSS are defined as the lower dimensional spaces when the number of non-zero parameters is less than 8. Previous work in this area has focused on understanding the structure of 8-dimensional QSS through 2- and 3-sections which led to many interesting geometric shapes. This approach could not be extended to 4 and higher order sections because of the difficulties in visualizing them. We show that it is possible to do just that by using some special 3-dimensional vectors representing the complexity of QSS.

3. New 3-Dimensional Qutrit State Vectors

The most general Qutrit Density Matrix (QDM) can be represented using the well-known SU (3) invariant form as $\rho = \frac{1}{3} I_3 + \vec{\boldsymbol{n}}.\vec{\boldsymbol{\lambda}}$. Here I_3 is 3×3 unit matrix, , $\vec{\mathbf{n}} = (n_1, n_2, \ldots, n_8)$ are 8 real parameters, and , $\vec{\boldsymbol{\lambda}} = (\lambda_1, \lambda_2, \ldots, \lambda_8)$ are 3×3 Gell-Mann matrices. An alternative representation of the QDM utilized the symmetric part of the 2 qubit Bloch matrix representing a spin-1 state.

$$\rho = \begin{bmatrix} \omega_1 & (q_3 + ia_3)/2 & (q_2 - ia_2)/2 \\ (q_3 - ia_3)/2 & \omega_2 & -(q_1 + ia_1)/2 \\ (q_2 + ia_2)/2 & -(q_1 - ia_1)/2 & \omega_3 \end{bmatrix}$$

The parameters in this representation are connected to spin-1 observables: $\omega_i = < S_i^2 >= Tr(\rho S_i^2)$, $a_i = < S_i >= Tr(\rho S_i)$, and $q_k = < S_i S_j + S_j S_i >= Tr\{\rho(S_i S_j + S_j S_i)\}, k \neq i, j$.

Comparison of the 2 parametrization of QDM with respect to the constraints imposed on QDM gives the first 2 constraints on the spin-1 based QDM parameters.

(i) $\omega_1 \geq 0, \omega_2 \geq 0, \omega_3 \geq 0$,

(ii) $Tr(\rho) = 1$ or $\omega_1 + \omega_2 + \omega_3 = 1$,

Using the definition $r_i^2 = \frac{1}{4}\left(a_i^2 + q_i^2\right)$, the other 2 constraints take the following form.

(i) $\sum_{i=1}^{3}\left(\omega_i^2 + 2r_i^2\right) \leq 1$ Implied by $Tr\ (\rho^2) \leq 1$, equality for pure states.

(ii) $\sum_{i=1}^{3}(3\omega_i^2 - 2\omega_i^3 + 6\omega_i r_i^2) - \frac{1}{4}(a_2 a_3 q_1 + a_3 a_1 q_2 + a_1 a_2 q_3 - q_1 q_2 q_3) \leq 1$

Implied by $\det \rho \geq 0$.

The above constraints can be rewritten in terms of the following 3-dimensional vectors.

(i) $\vec{\mathbf{u}} = \{u_1, u_2, u_3\} = \left\{\sqrt{\omega_1^2 + 2r_1^2}, \sqrt{\omega_2^2 + 2r_2^2}, \sqrt{\omega_3^2 + 2r_3^2}\ \right\}, u^2 \leq 1$

(ii) $\vec{\mathbf{v}} = \{v_1, v_2, v_3\}$, $v^2 \leq 1$ with

$$v_1 = \sqrt{\omega_1(1 - 2\omega_2\omega_3 + 6r_1^2) - \frac{1}{4}q_1\left(a_2 a_3 - \frac{1}{3}q_2 q_3\right)}$$

$$v_2 = \sqrt{\omega_2(1 - 2\omega_3\omega_1 + 6r_2^2) - \frac{1}{4}q_2\left(a_3 a_1 - \frac{1}{3}q_3 q_1\right)}$$

$$v_3 = \sqrt{\omega_3(1 - 2\omega_1\omega_2 + 6r_3^2) - \frac{1}{4}q_3\left(a_1 a_2 - \frac{1}{3}q_1 q_2\right)}$$

(iii) $\vec{\mathbf{w}} = \{\sqrt{\omega_1},\ \ \sqrt{\omega_2},\ \ \sqrt{\omega_3}\}, w^2 = \omega_1 + \omega_2 + \omega_3 = 1$

These vectors are always in the positive octant of a 3-dimensional sphere of unit radius. The mixed states reside inside the volume interior to the “pure qutrit state surface”. This is similar to the Bloch ball of a qubit.

4. 2- and 3-Sections of the Qutrit State Space

In the 8-dimensional QSS, visualization of state space is very difficult except in 2 and 3 dimensions. In reference [2, 3] those slices were derived and shown to have interesting geometrical shapes. Any attempts to go beyond the 3-dimensions proved almost impossible. The new vector representation is free of this problem and sections of all possible dimensions can be understood as affecting the three vectors in specific manner.

As an example, we will present the 2-sections in their terms to illustrate this point. The cubic constraint $\det\rho \geq 0$ based on Gell-Mann representation is given as

$$3\sum_{i=1}^{8} n_i^2 - 6n_8\left(\sum_{i=1}^{3} n_i^2 - \frac{1}{2}\sum_{i=4}^{7} n_i^2 - \frac{1}{3}n_8^2\right) - 6\sqrt{3}(n_1n_4n_6 + n_1n_5n_7 + n_2n_5n_6 - n_2n_4n_7)$$
$$-3\sqrt{3}n_3\left(n_4^2 + n_5^2 - n_6^2 - n_7^2\right) = 1$$

The number of 2-sections, when any of the two parameters are non-zero, is 28. They are distributed in 4 classes describing circle, triangle, parabola and ellipse shapes in the 8-dimensional QSS. As an example $\{n_1, n_2\}$ section leads to $n_1^2 + n_2^2 = \left(1/\sqrt{3}\right)^2$, which is the equation for a circle of radius $1/\sqrt{3}$. In terms of spin-1 QSS parameters we get $q_3^2 + a_3^2 = (2/3)^2$. In terms of the 3-dimensional vectors, one has $u^2 = \frac{1}{3} + (q_3^2 + a_3^2)/2 \leq \frac{5}{9}$, $v^2 = \frac{7}{9} + \frac{1}{2}(q_3^2 + a_3^2) \leq 1$, and $w^2 = 1$. The states are not pure because $u^2 < 1$. So the $\{n_1, n_2\}$ section for a circle translates into constraints on the lengths of $\vec{\mathbf{u}}$, and $\vec{\mathbf{v}}$ vectors.

Similarly other 2-sections, 3-sections, and higher-order sections lead to corresponding expressions for $\vec{\mathbf{u}}$, $\vec{\mathbf{v}}$, and $\vec{\mathbf{w}}$-vector lengths and individual components . This will lead to a catalog of all the higher-order sections in terms of these vector expressions. The formalism can be extended to the 2-qutrit entangled states as well.

5. Conclusions

In this work we have shown that the 8-dimensional QSS can be alternatively visualized in 3 dimensions using 3-dimensional vectors. It is seen that 2 and other higher order QSS sections can be expressed as constraints on the vectors. Implications of these constraints for higher order sections (e.g., 3-7 sections) will be presented in a systematic way in future.

6. REFERENCES

[1] Vinod K. Mishra, "Three Dimensional Visualization of Qutrit States", arXiv:1611.02701

[2] G. Sarbicki and I. Bengtsson, "Dissecting the Qutrit ", J. Phys. A: Math. Theor. 46, 035306 (2013)

[3] Istok P. Mendas, "The classification of three-parameter density matrices for a qutrit", J.Phys. A: Math. Gen. 39 (2006) 11313-11324

[4] S. K. Goyal, B. N. Simon, R. Singh, and S. Simon, "Geometry of the generalized Bloch sphere for qutrits ", arXiv: 1111.4427

[5] Ingemar Bengtsson, Stephan Weis and Karol Zyczkowski, "Geometry of the set of mixed quantum

states: An apophatic approach“, arXiv: 1111.4427v2

[6] Pawel Kurzynski, Adrian Kolodziejski, Wieslaw Laskowski, and Marein Markiewicz, “Three-dimensional visualization of a qutrit”, arXiv:1601.07361v1

[7] Pawel Kurzynski, “Multi-Bloch Vector Representation of the Qutrit”, arXiv: 0912.3155v1