Efficient Simulation of Random States and Random Unitaries

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QCrypt 2020, in Cyberspace

Joint Center for Quantum Information and Computer Science

QuSoft Research Center for Quantum Software

UCONN

Results — overview

- We study the **simulation of random quantum objects**, i.e. random quantum states and random unitary operations
- We develop a **theory of** their **stateful simulation**, a quantum analogue of “lazy sampling”
- For random states, we develop an efficient protocol for stateful simulation
- For random unitaries, we show that simulation can be done in polynomial space
- As an **application**, we design a **quantum money** scheme that is unconditionally unforgeable and untraceable.

Introduction

Randomness…

…is extremely useful. Applications:

- All of cryptography
- Monte Carlo simulation
- Randomized algorithms
- …

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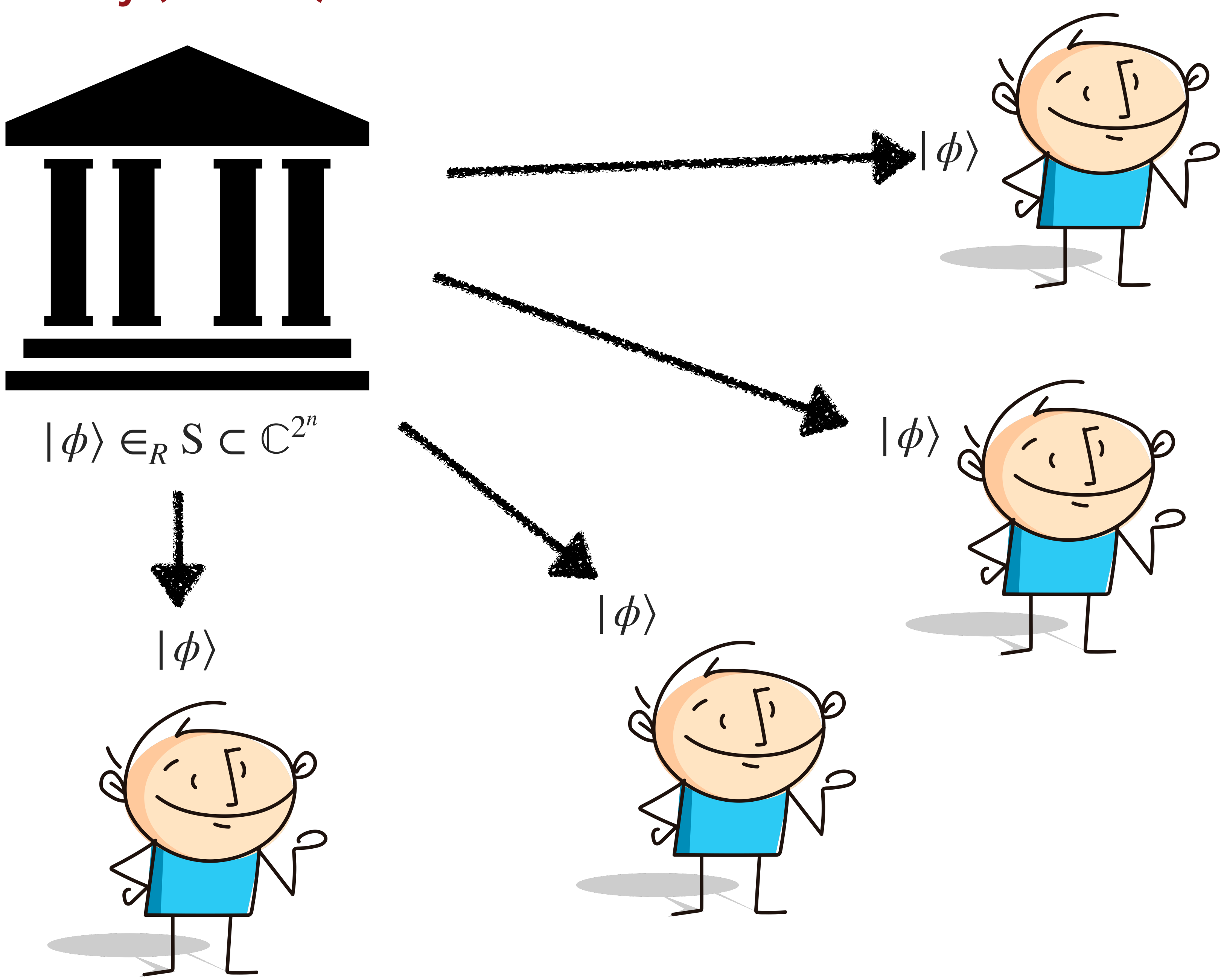

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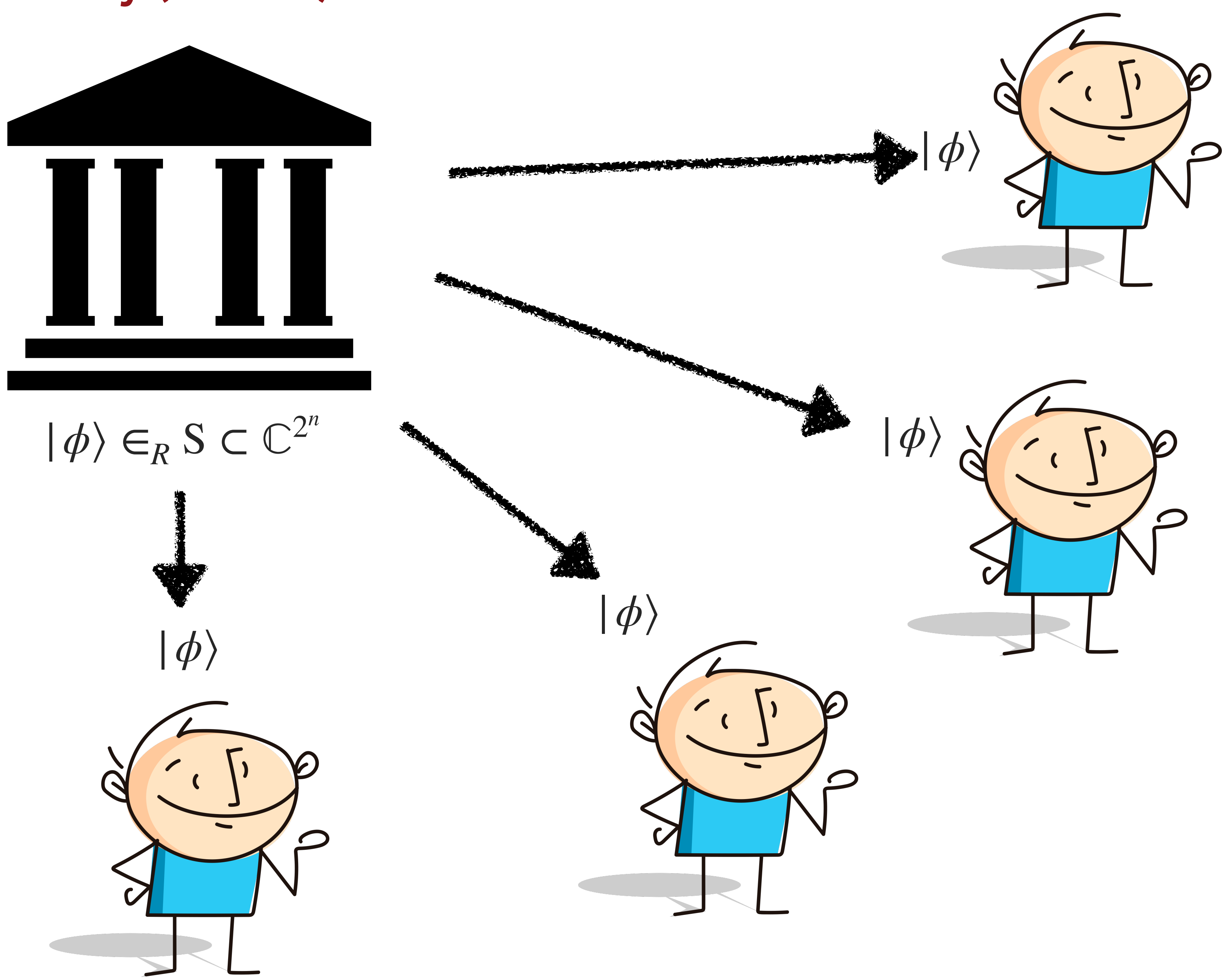

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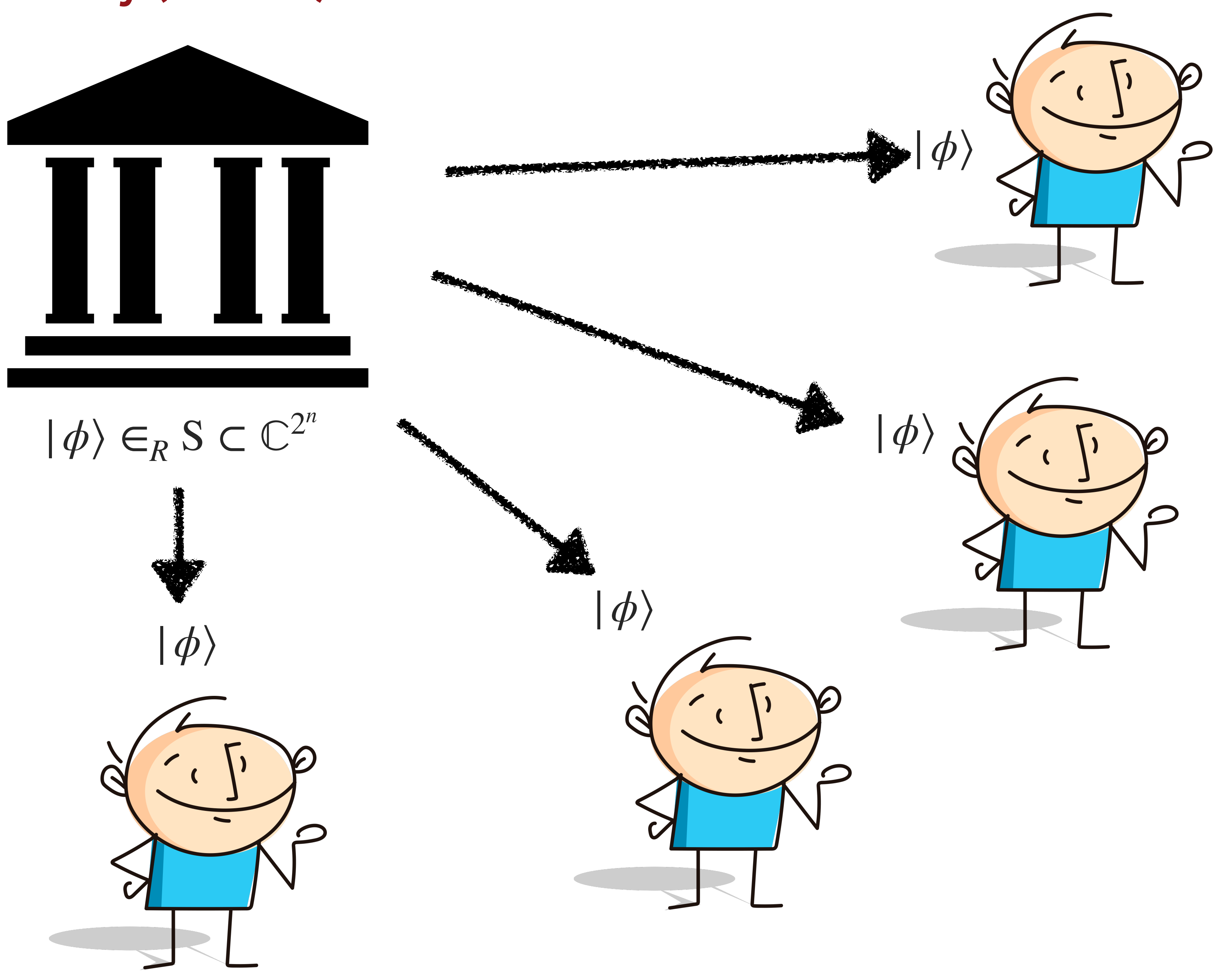

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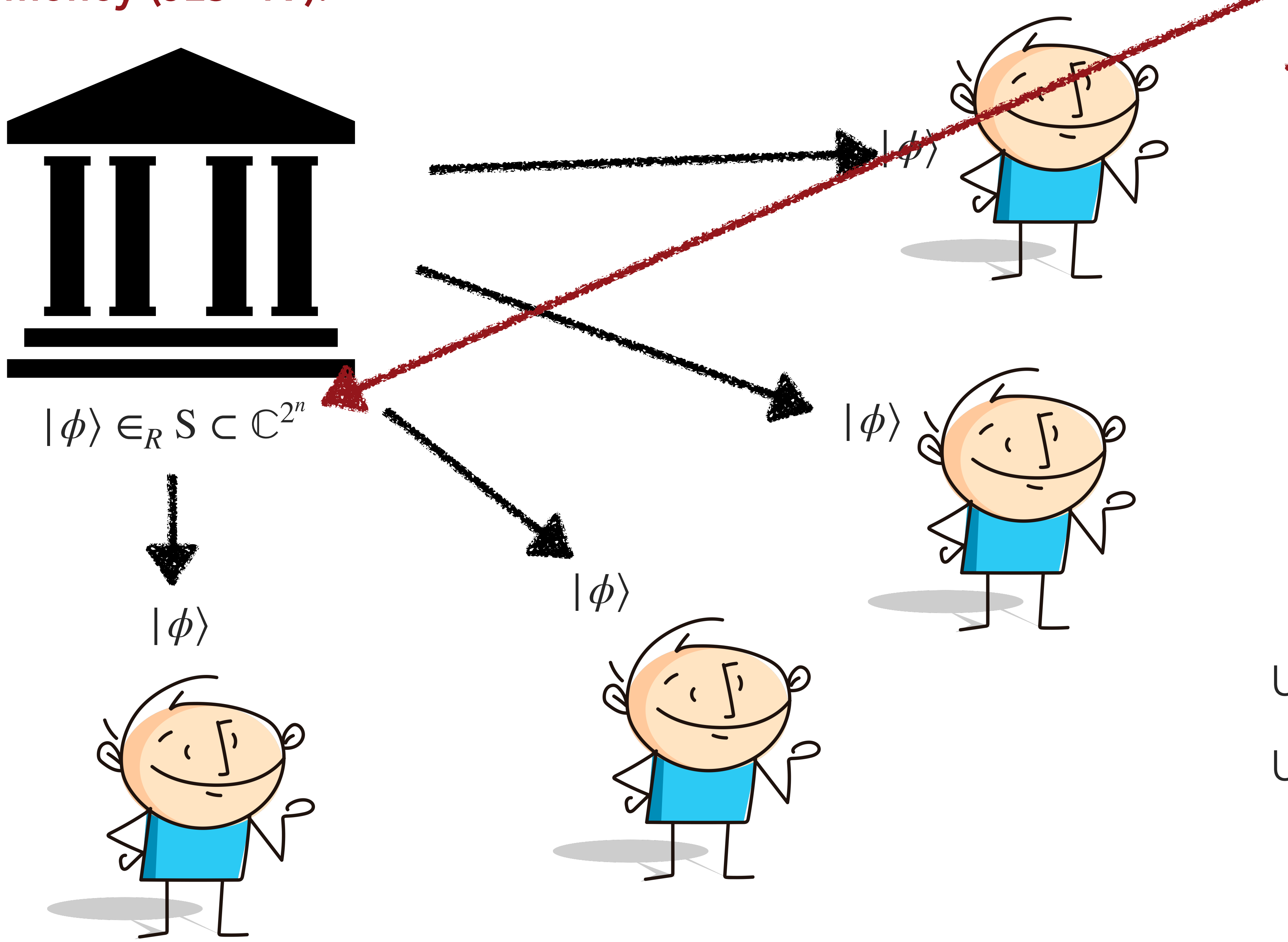

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Simulation of random quantum objects

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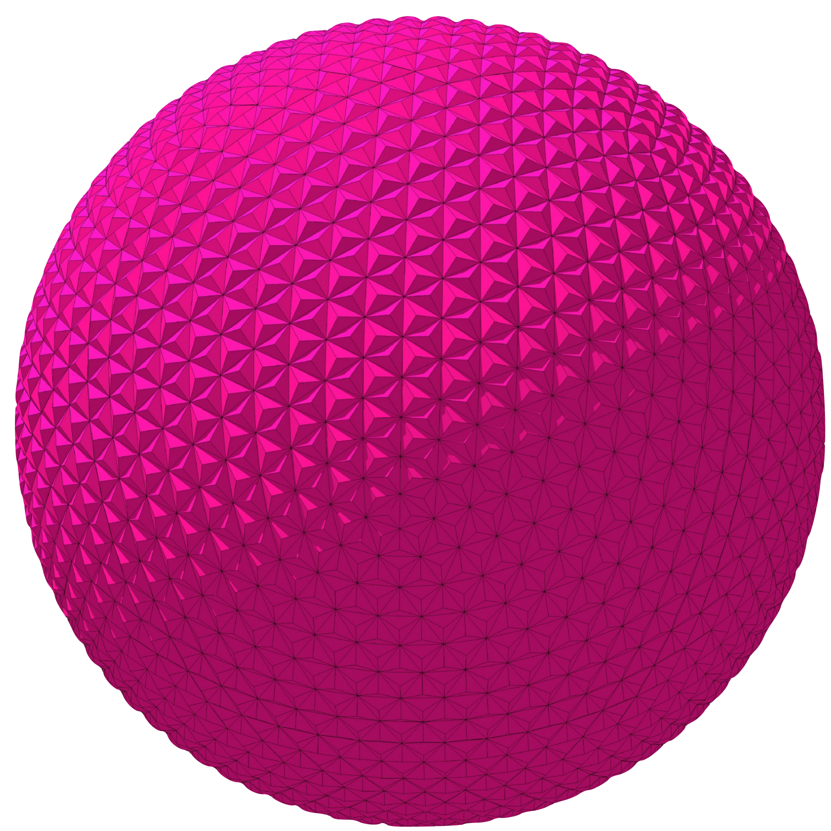

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- **Use stateful simulation, unconditionally secure untraceable quantum money (AMR)**

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Similar argument for unitaries.

Techniques

Going to both churches…

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From representation theory: $\mathbb{E}_{|\psi\rangle \sim \mathrm{Haar}}\left[|\psi\rangle\langle\psi|^{\otimes \ell} \right] = \tau_{\mathrm{Sym}^{\ell}\mathbb{C}^d}$

Stateful simulation algorithm

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Strategy:

1. Maintain maximally entangled state of two copies of $\mathrm{Sym}_{d,\ell}$.
2. On query: extend it from ℓ to $\ell + 1$ by acting on one of the copies only.

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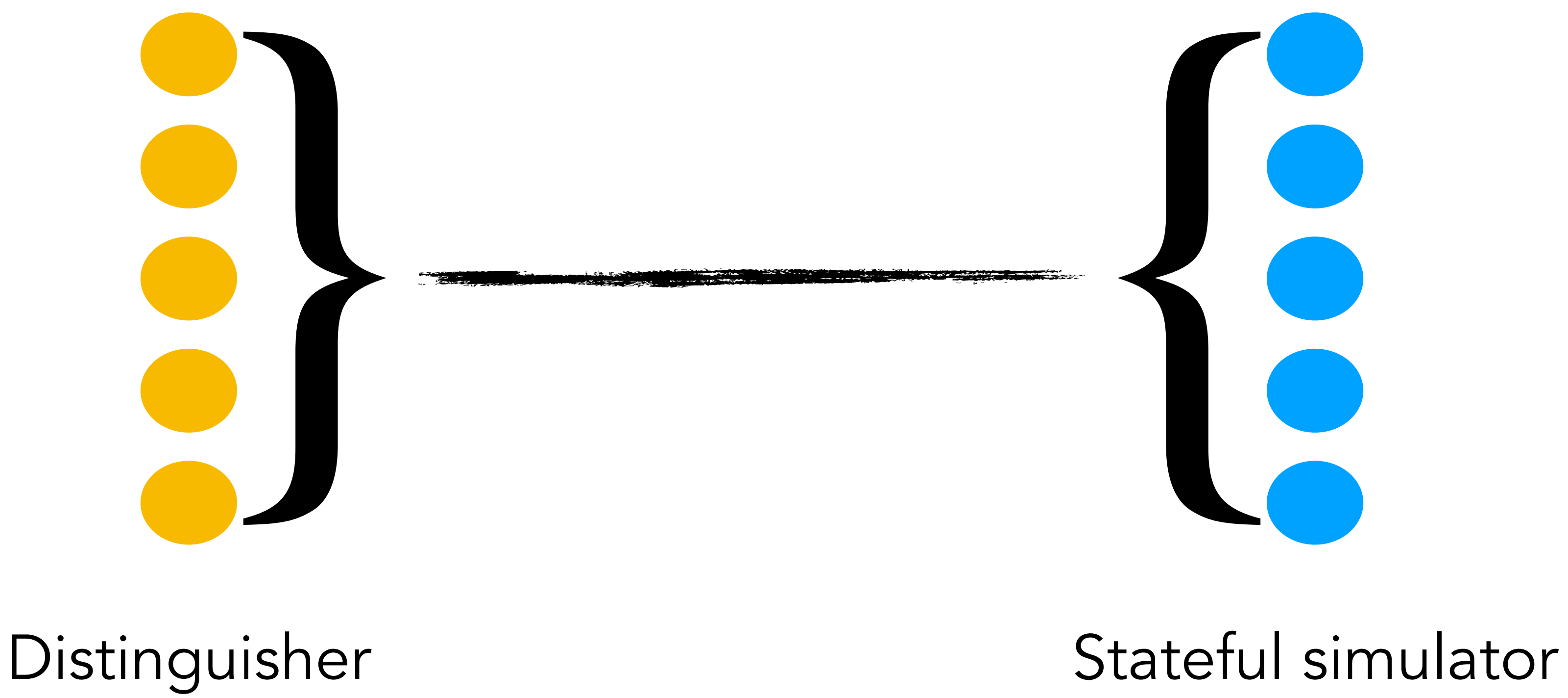

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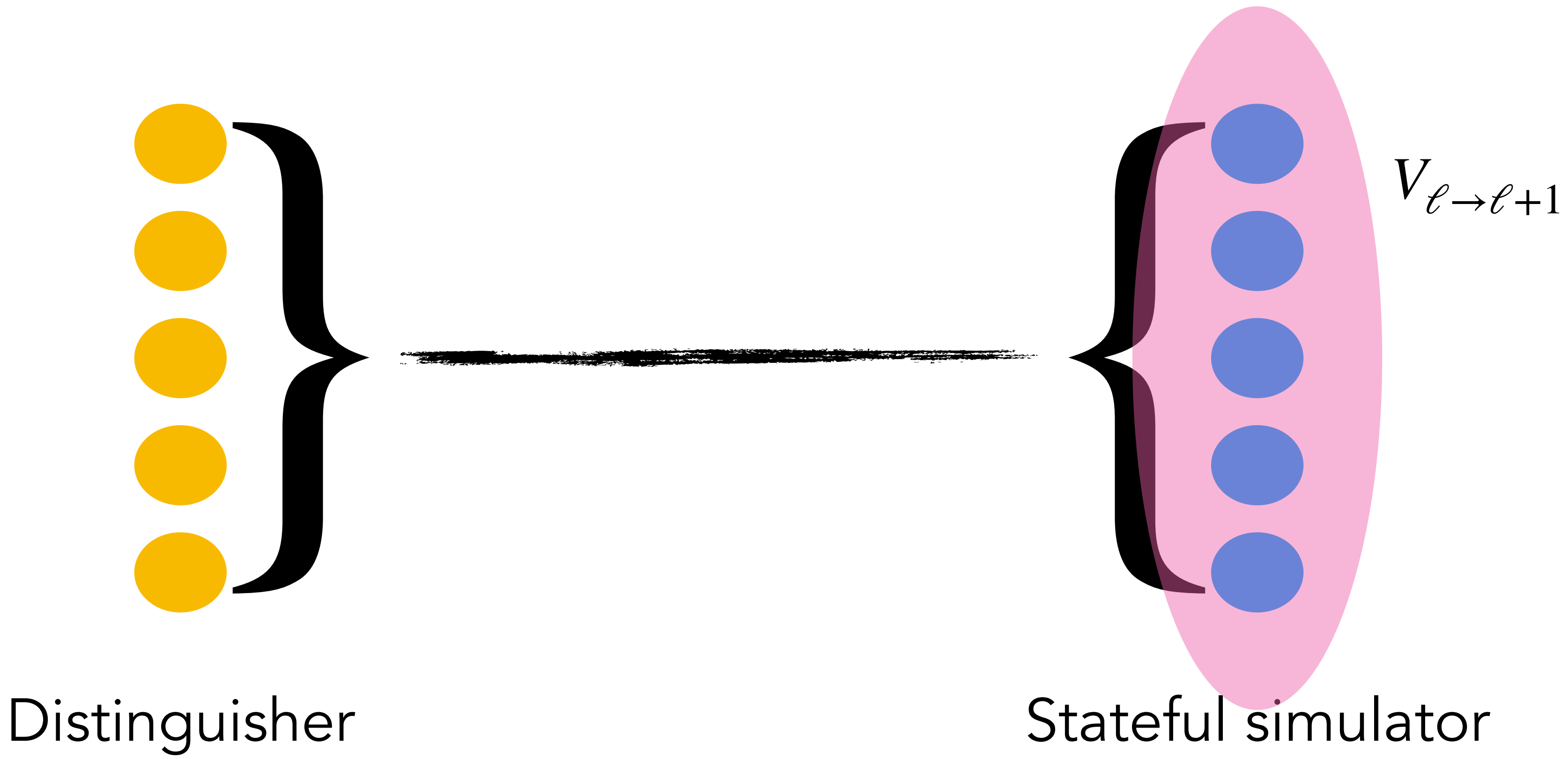

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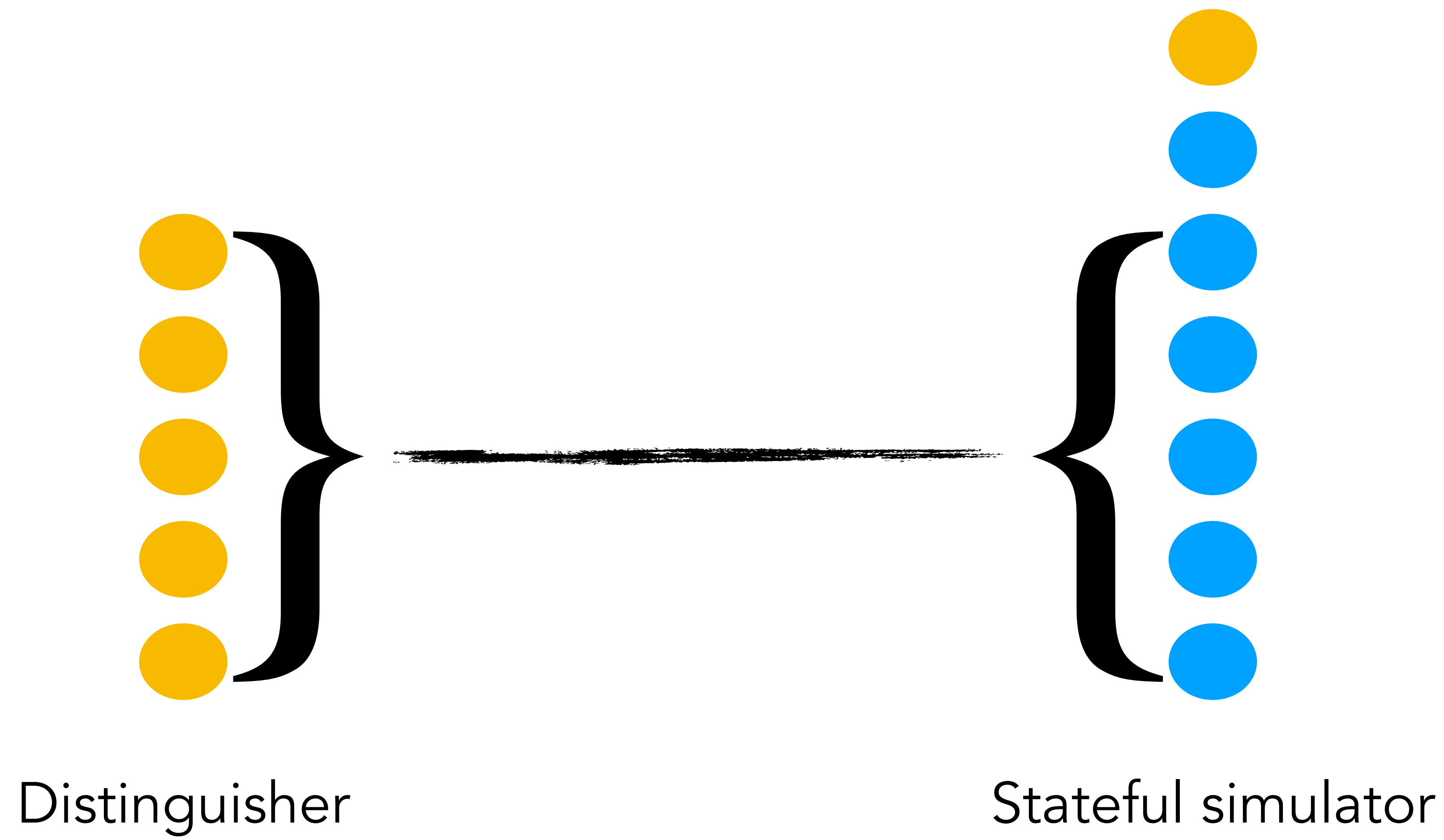

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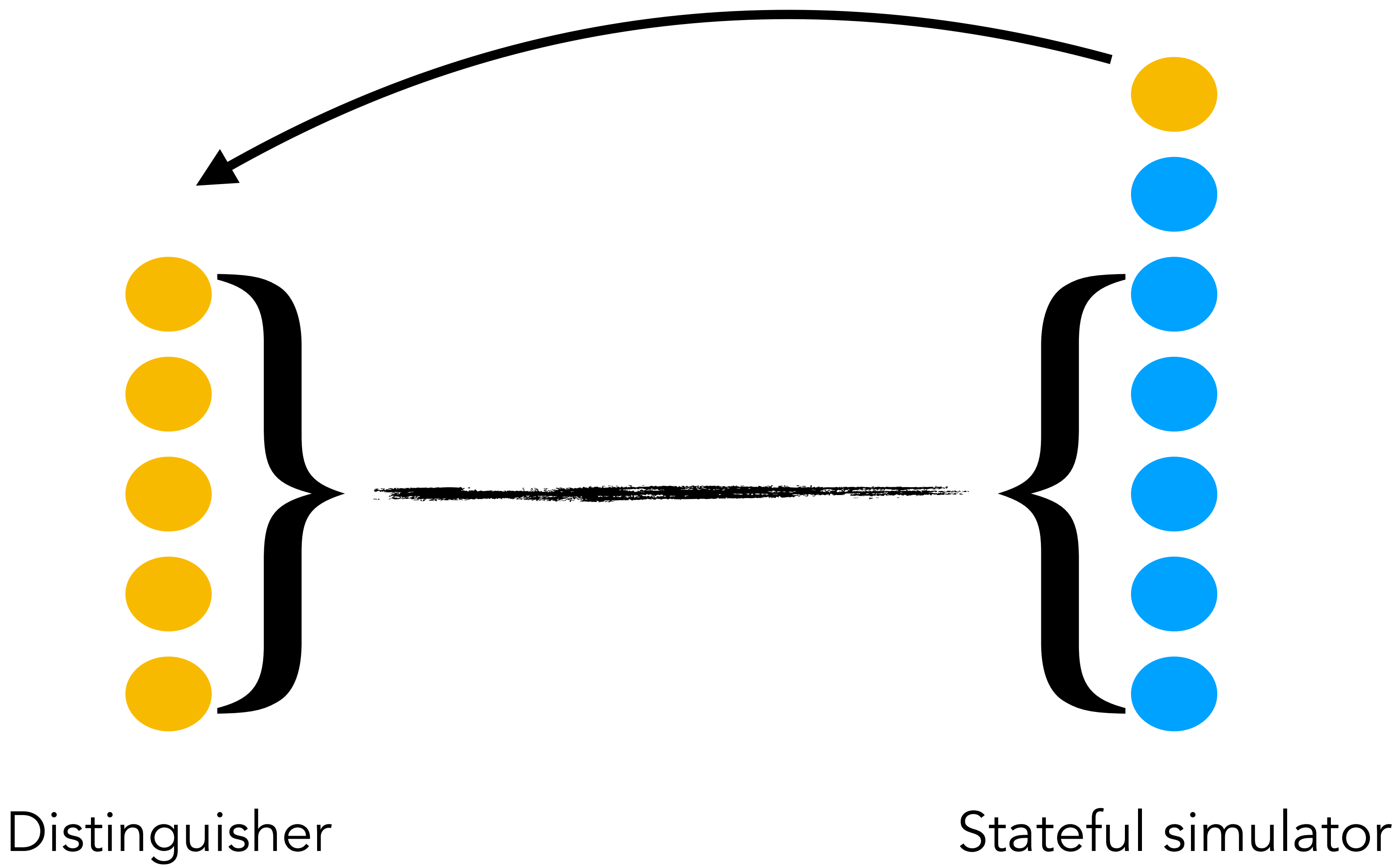

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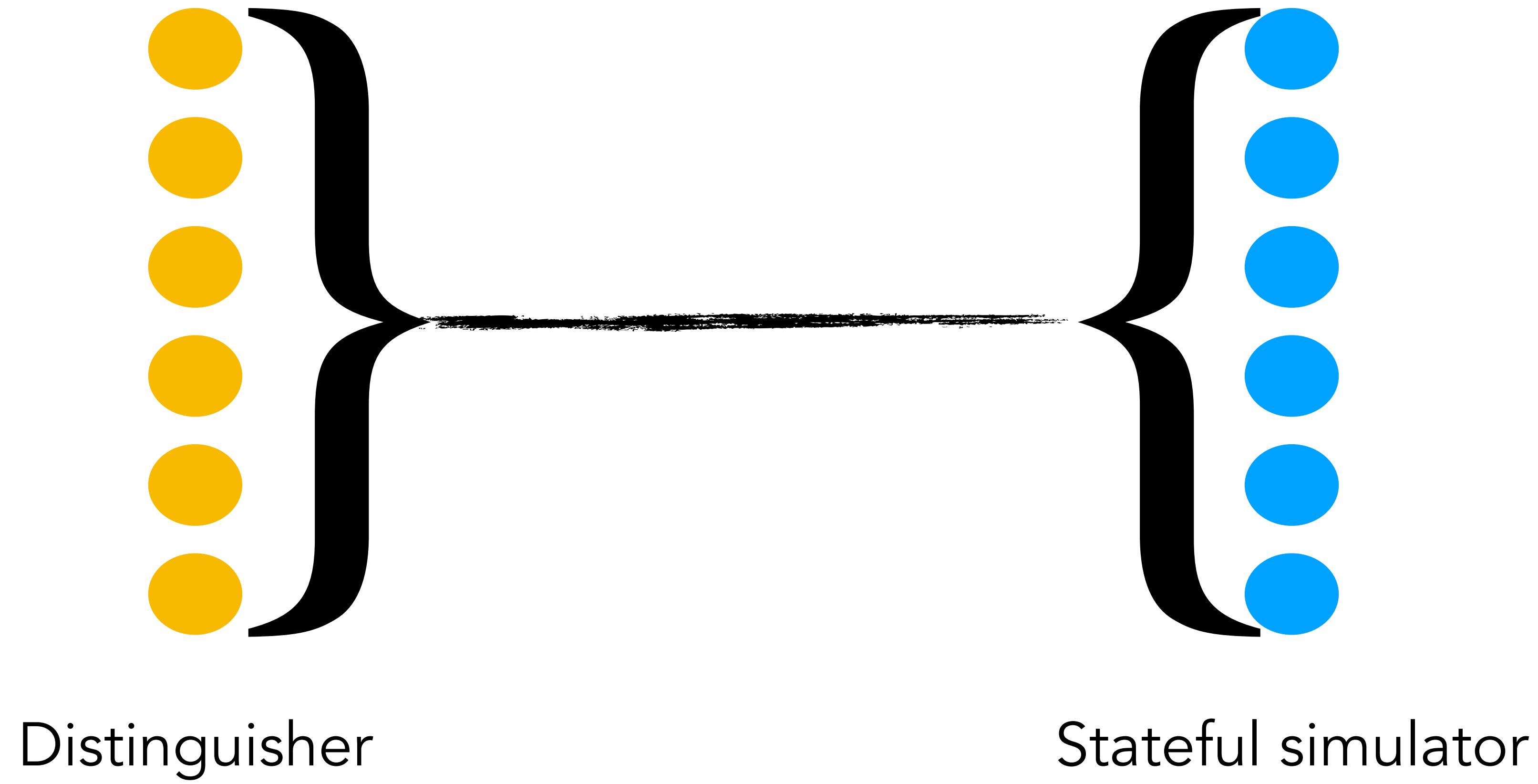

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  - Uniqueness property of the Stinespring dilation

Summary, open questions

Summary:

- We develop a theory of stateful simulation of random quantum primitives.
- Random quantum states can be approximately simulated efficiently using a stateful algorithm
- Random unitaries can be simulated exactly in a space-efficient way using a stateful algorithm.
- The random state simulator can be used to construct unconditionally secure untraceable quantum money.

Open questions:

- Can we simulate random unitaries efficiently?
- (From JLS '19) Construct pseudorandom unitaries!