Thirty six entangled officers of Euler

Quantum solution to a classically impossible problem

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Abstract

The negative solution to the famous problem of 36 officers of Euler implies that there are no two orthogonal Latin squares of order six. We show that the problem has a solution, provided the officers are entangled, and construct orthogonal quantum Latin squares of this size. As a consequence, we find an example of the long elusive Absolutely Maximally Entangled state AME$(4,6)$ of four subsystems with six levels each, equivalently a 2-unitary matrix of size 36, which maximizes the entangling power among all bipartite unitary gates of this dimension, or a perfect tensor with four indices, each running from one to six. This result allows us to construct a pure nonadditive quhex quantum error detection code $((3,6,2))_6$, which saturates the Singleton bound and allows one to encode a 6-level state into a triplet of such states.

Absolutely maximally entangled (AME) states

A pure state of n qudits (d level systems) is called *absolutely maximally entangled* if it is maximally entangled along any bipartition of size $\lfloor n/2 \rfloor$ and is denoted as AME(n,d) [1]. Examples are the Bell state $(|00\rangle + |11\rangle)/\sqrt{2}$ and the GHZ state $(|000\rangle + |111\rangle)/\sqrt{2}$.

- AME(n,d) is said to be of *minimal support* if its expansion in the product basis contains $d^{\lfloor N/2 \rfloor}$ non-zero coefficients all being equal to $1/\sqrt{d^{\lfloor N/2 \rfloor}}$.
- Generalized Bell state: $|\phi^+\rangle = \sum_{i=1}^{d} |ii\rangle/\sqrt{d}$ is an example of AME$(2,d)$ with minimal support.
- Existence of AME(n,d) for general n, d is a non-trivial problem and answer is not known.
- For qubits, AME$(n, d=2)$ exist only for $n = 2, 3, 5, 6$.

Existence of AME state of four quhex systems; AME$(4,6)$, was an open problem.

Correspondence between AME states and unitary operators

- We consider n to be even. In particular, we will be interested in $n = 4$.
- Even party AME states are in one-to-one correspondence with threshold quantum secret sharing (QSS) schemes [1].
- $n = 2$ case: Bipartite pure state in $\mathcal{H}_d^A \otimes \mathcal{H}_d^B$,

$$|\psi\rangle_{AB} = (C \otimes \mathbb{I})|\phi^+\rangle_{AB}, \quad |\phi^+\rangle_{AB} = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} |i\rangle_A \otimes |i\rangle_B$$

is maximally entangled iff $U = C/\sqrt{d}$ is unitary; $U \in \mathbb{U}(d) \;\Leftrightarrow\;$ AME$(2,d)$.

- $n = 4$ case:
A pure state in $\mathcal{H}_d^A \otimes \mathcal{H}_d^B \otimes \mathcal{H}_d^C \otimes \mathcal{H}_d^D$,

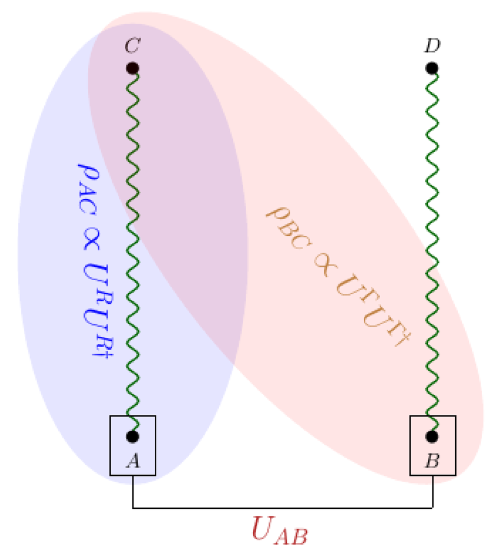
$$|\psi\rangle_{ABCD} = (C_{AB} \otimes \mathbb{I})|\phi^+\rangle_{AC} \otimes |\phi^+\rangle_{BD},$$

is maximally entangled with respect to:

(a) AB/CD partition if U is unitary.

(b) AC/BD partition if U^R is unitary, where $\langle i\alpha|U^R|j\beta\rangle = \langle ij|U|\alpha\beta\rangle$.

(c) AD/BC partition if U^Γ is unitary, where $\langle i\alpha|U^\Gamma|j\beta\rangle = \langle i\beta|U|j\alpha\rangle$.



- Unitary U for which both such rearrangements are unitary i.e, U, U^R, $U^\Gamma \in \mathbb{U}(d^2)$ is called *2-unitary* and corresponds to AME$(4,d)$ [2]. In general,

$$\text{AME}(2k, d) \Leftrightarrow k\text{-unitary}.$$

- 2-unitary operators are *maximally entangling* [3].

Connection between AME states and error correcting codes

- Measurement of any local observable in an AME state is completely random.
- This property makes AME states useful for error correction. For example, the AME state of four qutrits given by

$$|\text{AME}(4,3)\rangle = \frac{1}{3}[|1111\rangle + |1223\rangle + |1332\rangle + |2122\rangle + |2231\rangle + |2313\rangle + |3133\rangle + |3212\rangle + |3321\rangle],$$

can correct any single qutrit error. Note that the Hamming distance between different terms is 3.

- Minimal support AME $\Leftrightarrow$ Maximum Distance Separable (MDS) codes [2].

Connection between AME states and combinatorics

- Description of multipartite entanglement in terms of combinatorial structures is powerful.
- A particular class of AME$(4,d)$ states are in one-to-one correspondence to *orthogonal Latin squares* (OLS). For example, $|\text{AME}(4,3)\rangle$ gives a pair of OLS.

$$\{1,2,3\} \times \{1,2,3\} \longrightarrow \text{OLS}(3): \begin{array}{|c|c|c|} \hline 11 & 23 & 32 \\ \hline 22 & 31 & 13 \\ \hline 33 & 12 & 21 \\ \hline \end{array} = \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 2 & 3 & 1 \\ \hline 3 & 1 & 2 \\ \hline \end{array} \cup \begin{array}{|c|c|c|} \hline 1 & 3 & 2 \\ \hline 2 & 1 & 3 \\ \hline 3 & 2 & 1 \\ \hline \end{array}.$$

- OLS$(d) \Leftrightarrow$ 2-unitary permutation $\Leftrightarrow$ AME$(4,d)$ (minimal support).

Problem of 36 officers of Euler

"Six different regiments have six officers, each one belonging to different ranks. Can these 36 officers be arranged in a square formation so that each row and column contains one officer of each rank and one of each regiment?"

- Solution to this problem lies in the existence of OLS(6).
- Non-existence of OLS(6): Conjectured by Euler (1779) proved by Gaston Tarry (1901).
- OLS(d) exist $\forall\, d$ except for $d = 2, 6$ [4].
- Combinatorial structures closest to being OLS(6) [5]: **Quasi-OLS of order six**

$$\{A, K, Q, J, 10, 9\} \times \{\spadesuit, \clubsuit, \diamondsuit, \heartsuit, ✿, ✷\}$$

A♠	K♣	Q♦	J♥	10✿	9✷
K♦	A♥	J✿	Q✷	9♠	10♣
Q♣	J♠	9♥	10♦	A✷	K✿
J✷	Q✿	10♠	9♣	K♥	A♦
10♥	9♦	K✷	A✿	J♣	Q♠
9✿	10✷	A♣	K♠	Q♦	J♥

- **There is no AME$(4,6)$ state of minimal support.**
- Existence of AME$(4,6)$ with larger support cannot be ruled out from this negative result.

Search for the quantum solution

- Promoting each symbol in LS to a quantum state: *Quantum Latin square* (QLS) [6].
- Each row and column in QLS forms an orthonormal basis in $\mathcal{H}_d$.
- Orthogonal quantum Latin square (OQLS): Pair of QLS which form orthonomal basis in $\mathcal{H}_d \otimes \mathcal{H}_d$.

$$\text{OQLS of order 3: } \begin{array}{|c|c|c|} \hline |1\rangle & |2\rangle & |3\rangle \\ \hline |3\rangle & |1\rangle & |2\rangle \\ \hline |2\rangle & |3\rangle & |1\rangle \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline b|1\rangle + a|3\rangle & -a|1\rangle + b|3\rangle & |2\rangle \\ \hline a|1\rangle - b|3\rangle & |2\rangle & b|1\rangle + a|3\rangle \\ \hline |2\rangle & b|1\rangle + a|3\rangle & -a|1\rangle + b|3\rangle \\ \hline \end{array},$$

where $a = \sin(\pi/6) = 1/2$, $b = \cos(\pi/6) = \sqrt{3}/2$.

OQLS of order 6: The genuine quantum solution

- Does OQLS of order six or, 2-unitary unitary matrix (with complex entries) of order 36 exist?
- Using non-linear maps introduced in [3], we show that 2-unitaries of order 36 exists!
- Using appropriate local rotations a nice form of 2-unitary matrix can be obtained:

$$\mathcal{U}_{36} = (U_1 \otimes U_2) U_{\text{num}} (U_3 \otimes U_4); \; U_i \in \mathbb{U}(6).$$

- Corresponding combinatorial design is entangled and hence does not separate into a pair QLS.
- The problem of 36 officers by Euler in quantum case has a solution: Ranks and regiments of the officers are entangled.

|A♣⟩ |K♠⟩	|A♦⟩ |K♥⟩	|10✿⟩ |10✷⟩ |9✿⟩ |9✷⟩	|10♠⟩ |10♣⟩ |9♠⟩ |9♣⟩	|Q♦⟩ |Q♥⟩ |J♦⟩ |J♥⟩	|Q✿⟩ |Q✷⟩ |J✿⟩ |J✷⟩
|10♣⟩ |9♠⟩	|10♦⟩ |9♥⟩	|Q✿⟩ |Q✷⟩ |J✿⟩ |J✷⟩	|Q♠⟩ |J♣⟩	|A♦⟩ |A♥⟩ |K♦⟩ |K♥⟩	|A✿⟩ |A✷⟩ |K✿⟩ |K✷⟩
|Q✿⟩ |J✷⟩	|Q♣⟩ |J♠⟩	|A♥⟩ |K♦⟩	|A✿⟩ |A✷⟩ |K✿⟩ |K✷⟩	|10♠⟩ |9♣⟩	|10♦⟩ |10♥⟩ |9♦⟩ |9♥⟩
|A✿⟩ |A✷⟩ |K✿⟩ |K✷⟩	|A♠⟩ |A♣⟩ |K♠⟩ |K♣⟩	|10♦⟩ |10♥⟩ |9♦⟩ |9♥⟩	|10✿⟩ |10✷⟩ |9✿⟩ |9✷⟩	|Q♠⟩ |Q♣⟩ |J♠⟩ |J♣⟩	|Q♦⟩ |Q♥⟩ |J♦⟩ |J♥⟩
|10♥⟩ |9♦⟩	|10✷⟩ |9✿⟩	|Q♠⟩ |Q♣⟩ |J♠⟩ |J♣⟩	|Q♦⟩ |J♥⟩	|A✿⟩ |A✷⟩ |K✿⟩ |K✷⟩	|A♠⟩ |K♣⟩
|Q♥⟩ |J♦⟩	|Q✷⟩ |J✿⟩	|A♠⟩ |A♣⟩ |K♠⟩ |K♣⟩	|A♦⟩ |A♥⟩ |K♦⟩ |K♥⟩	|10✿⟩ |9✷⟩	|10♠⟩ |10♣⟩ |9♠⟩ |9♣⟩

- If there was only one term in each cell: No entanglement implies a pair of QLS.
- Surprisingly each cell is like a two-qubit Bell state (either with support 2 or 4).

Conclusion

- We have addressed the open problem about the existence of AME state of four quhex systems; AME$(4,6)$, positively.
- Classical OLS of order six (2-unitary permutation of order 36): classical solution, does not exist but *entangled* OQLS of order six (2-unitary unitary matrix of order 36): quantum solution, exists.

Future directions

- Existence of other classes; possibly simpler and different entanglement, of AME states in $d = 6$.
- Existence of *genuinely quantum OQLS* in other dimensions.
- AME states with more than 4 parties like eight party state of ququarts; AME$(8,4)$, whose existence is also open till now!

e-print is available at arXiv:2104.05122

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